

## **How Much Confidence Do We Have in Estimates of Future Net Discount Rates?**

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### **I. Introduction**

A recent survey by Brookshire, et. al., (2006) suggests that forensic economists are mixed in their use of current versus historical data to estimate future net discount rates.<sup>1</sup> Typically the choice is between a long-term average or simply the current prevailing rate.<sup>2</sup> Cushing and Rosenbaum (2006) proposed an estimator that optimally uses the information in the past behavior of net discount rates to forecast future rates. Their optimal estimator is a weighted average of the current and long-term mean net discount rates with weights that depend on the degree of persistence in the net discount rate process and on the horizon over which one wishes to forecast future rates. Noting the practical impediments to calculating such an estimator, they proposed an alternate compromise estimator that equally weights the current net discount rate with a long-term average net discount rate.

In this paper we extend that analysis by developing confidence intervals for the current, long-term average, optimal and compromise estimators. In particular, we develop both analytic and bootstrap estimates of 50% confidence intervals for all four estimators. The 50% boundary demarcates values that are more likely than not to occur. This serves two purposes. One is to shed light on error rates as is called for in Daubert (1993). The other is to help establish a reasonable degree of economic certainty in our net discount rate predictions.

Results show that the boundaries vary by estimator and by forecast horizon. However, in almost all cases, the boundaries are within two percentage points of the point estimates. These boundaries generally narrow as the horizon increases. For the optimal and compromise estimators, the boundaries approach the point estimate plus or minus about one percentage point for a 20-year horizon and beyond. Additional results show that the boundaries have remained quite stable over time, regardless of the estimator. Combining these results with earlier findings, the best range of predictions for the future net

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<sup>1</sup>In their most recent survey, Brookshire, et al., report that 41% of respondents use historical interest rates while 34% use current interest rates.

<sup>2</sup>For a sample of applications and analyses, see Braun and Strazicich (2005); Cushing and Rosenbaum (2006); Ewing, Payne and Piette (2001, 2003); Gamber and Sorensen (1993, 1994); Haslag, Nieswiadomy and Slottje (1991, 1994); Hays, Schreiber, Payne, Ewing and Piette (2000); Horvath and Sattler (1997); Johnson and Gelles (1996); Payne, Ewing and Piette (1999a, 1999b); Sen, Gelles and Johnson (2000, 2002).

discount rate may be an equal weighting of the current rate and the long-term average, plus or minus one or two percentage points.

In section II, we develop a method to derive the confidence intervals. Section III discusses data and presents results. It is followed by a conclusion in section IV.

## II. The Model

In order to proceed formally, we first must be precise about what we mean by “future net discount rates.” Forensic economic settings typically require a single value that summarizes the future course of net discount rates. The value is presumably some average of net discount rates that might be expected to prevail over the relevant forecast horizon. We therefore define the Future Net Discount Rate,  $FNDR_t$ , as a geometrically declining weighted average of future net discount rates over a horizon of  $m$  periods,

$$(1) \quad FNDR_{t,m} = \frac{1-\gamma}{1-\gamma^m} \sum_{j=0}^{m-1} \gamma^j (ndr_{t+1+j}).$$

In equation (1),  $FNDR_{t,m}$  is the period  $t$  forecast of the future net discount rate projected out  $m$  years and  $ndr_{t+1+j}$  is the net discount rate in period  $t+1+j$ . The parameter  $\gamma$  governs the decay in the weighting scheme. For  $\gamma$  close to unity, future net discount rates are given equal weights.<sup>3</sup> For values of  $\gamma$  less than one, closer observations receive more weight than distant observations. As  $\gamma$  approaches zero, the weights become concentrated on the closest observation.<sup>4</sup> By analogy to the term structure of interest rates literature (see Shiller, 1979) we suggest choosing  $\gamma = 1/(1+ndr)$  where  $ndr$  is the long-run mean of the net discount rate process. In any case, our results are relatively insensitive to the choice of  $\gamma$  in the neighborhood of unity.

We consider a class of net discount rate estimators that take the form,

$$(2) \quad \widehat{PNDR}_t = \theta ndr_t + (1-\theta)\mu$$

where  $\widehat{PNDR}_t$  is the period  $t$  prediction of the net discount rate,  $\theta$  is the weight attached to the current observation and  $(1-\theta)$  is the weight attached to the unconditional mean of the net discount rate series,  $\mu$ . With respect to the four specific estimators, the long-term average estimator takes  $\theta = 0$ , the unit root estimator takes  $\theta = 1$ , the “compromise” estimator takes  $\theta = .5$  and the optimal estimator selects  $\theta$  to minimize a mean squared prediction error.<sup>5</sup>

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<sup>3</sup>Formally, the weight on  $j$ th term in equation (1) is  $\frac{(1-\gamma)\gamma^j}{1-\gamma^m}$ . Using L'Hospital's Rule, we can evaluate the limit as the parameter  $\gamma$  approaches unity,  $\lim_{\gamma \rightarrow 1} \frac{(1-\gamma)\gamma^j}{1-\gamma^m} = \lim_{\gamma \rightarrow 1} \frac{(1-\gamma)j\gamma^{j-1} - \gamma^j}{-m\gamma^{m-1}} = \frac{1}{m}$ . This corresponds to an equally weighted average.

<sup>4</sup>Formally, the value of the expression in equation (1) as  $\gamma$  approaches zero depends on the indeterminate form,  $0^0$ . However, if we adopt the common convention of defining  $0^0$  as unity, the statement in the text is exact.

<sup>5</sup>For a discussion on deriving the optimal  $\theta$ , see Cushing and Rosenbaum (2006).

The four estimators can each be viewed as weighted averages of the historical data. The long-term average places equal weight on all historical values whereas the random walk estimator places all weight on the current value. The compromise estimator, being an equal weighting of the random walk and long-term average places half of its weight on the current value and the rest equally on all of the previous observations. The optimal estimator is also a weighted average of the random walk and long-term average estimators, but the weights are determined by the time-series properties of the net discount rate.

Cushing and Rosenbaum (2006) evaluated the performance of the four estimators. They showed that from a theoretical standpoint, the optimal estimator was more efficient than either the long-term average or the current net discount rate. Benchmark estimates suggested that the forecast error variance using the optimal estimator was less than half that of the long-term average or current net discount rate. Examples from historical U.S. data and from recent international data showed that the optimal estimator would have performed better than either of the extreme alternatives.

In their previous paper, Cushing and Rosenbaum (2006) also suggested that calculating and justifying the particular weighting formula in the optimal estimator might be problematic in a forensic setting. Therefore, they also considered a compromise estimator that equally weighted the current and long-term average net discount rates. The form of the optimal estimator and the time-series properties of net discount rates suggested that this compromise estimator was likely to be reasonably close to the efficient solution. Empirical examples showed that the compromise estimator performed well, significantly outperforming both the long-term average and the current value. They also noted that while the compromise estimator should be asymptotically less efficient than the optimal estimator, it had the advantage of simplicity and reflected the essential concept of the optimal estimator: a blending of the two extreme estimators. Further, the theoretical considerations developed in that paper suggested that an equally weighted average may be close to optimal.

In this paper we extend their analysis by developing 50% confidence intervals for each of the estimators. Assuming normality, analytic confidence intervals for each of the estimators can be computed from:

$$(3) \quad \widehat{PNDR}_t \pm z_{.25} \sqrt{E(PNDR_t - \widehat{PNDR}_t)^2} .$$

From a sample of annual historical net discount rates it is possible to calculate the net discount rate in each year, as well as each of the four estimators. Sample variances can then be calculated and used to determine the  $z$  statistic for a 50% confidence interval.

Standard criteria (Akaike, Hannan-Quinn, and Schwartz) suggest that the data follow a first order autoregressive process.<sup>6</sup> We thus assume  $ndr_t$  follows a stationary first-order autoregressive process with AR parameter,  $\rho$ :

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<sup>6</sup>See Greene (2003), chapter 19.

$$(4) \quad ndr_t = \alpha + \rho ndr_{t-1} + e_t.$$

With this process for net discount rates, the mean squared prediction error of the estimators can be shown to be:<sup>7</sup>

$$(5) \quad E(PNDR_t - \widehat{PNDR}_t)^2 = \sigma_e^2 \frac{1}{(1-\rho^2)} \theta^2 - \sigma_e^2 \frac{2\rho(1-\gamma)(1-(\gamma\rho)^m)}{(1-\gamma\rho)(1-\rho^2)(1-\gamma^m)} \theta + \left[ \frac{(1-\gamma)^2 \sigma_e^2}{(1-\gamma^m)^2 (1-\rho^2)} \right] \left[ \left( \frac{(1+\gamma\rho)(1-\gamma^{2m})}{(1-\gamma\rho)(1-\gamma^2)} \right) - \left( \frac{2\gamma^m \rho^m}{1-\gamma\rho} \frac{1-(\gamma/\rho)^m}{1-(\gamma/\rho)} \right) \right].$$

The optimal estimator chooses  $\theta$  to minimize the above. The optimal choice for  $\theta$  is:

$$(6) \quad \theta^* = \frac{\rho(1-\gamma)(1-(\gamma\rho)^m)}{(1-\gamma\rho)(1-\gamma^m)}.$$

The analytic confidence intervals calculated from (3) and (5) suffer from two shortcomings. First, the expression for the mean squared prediction error, (5), assumes the parameters governing the net discount rate process,  $\rho$  and  $\alpha$  are known with certainty. The prediction intervals do not account for parameter uncertainty. Second, the confidence intervals assume the underlying disturbances in (4) are normally distributed. We can avoid both of these questionable assumptions by employing a bootstrap method to estimate confidence intervals.<sup>8</sup>

To apply the bootstrap, we estimate equation (4) to obtain estimates of the parameters of the net discount rate process,  $\hat{\alpha}$  and  $\hat{\rho}$ , and the residuals from the regression. Then we simulate an artificial net discount rate series of length  $n+m$  using the initial historic value, the estimated parameters and error terms drawn, with replacement, from the sample of residuals. The four net discount rate estimators specified in formula (2) above are then computed from the first  $n$  observations and the objective function in equation (1) is estimated from the remaining  $m$  terms.<sup>9</sup> The difference between each estimate and the computed objective function is then saved to develop an error. This procedure is repeated 5,000 times, yielding the sampling error distributions for each of the estimators. Bounds are then computed from each of the four sampling distributions by removing the top and bottom 25% of observations from each sampling error distribution. This gives us the 50% confidence intervals.

<sup>7</sup>For a derivation of equation (5), see the Appendix.

<sup>8</sup>For a more complete description of the bootstrap method, see Efron and Tibshirani (1993).

<sup>9</sup>We use a value of  $\gamma$  equal to 0.98.

Table 1  
Analytic 50% Confidence Bounds

Horizon (m)	Long-Term Average		Random Walk Estimator		Compromise Estimator		Optimal Estimator	
	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
1	-2.074	+2.074	-1.994	+1.994	-1.750	+1.750	-1.748	+1.748
5	-1.427	+1.427	-2.094	+2.094	-1.461	+1.461	-1.347	+1.347
10	-1.104	+1.104	-2.110	+2.110	-1.326	+1.326	-1.073	+1.073
20	-0.818	+0.818	-2.093	+2.093	-1.204	+1.204	-0.805	+0.805
35	-0.637	+0.637	-2.079	+2.079	-1.135	+1.135	-0.630	+0.630
50	-0.546	+0.546	-2.072	+2.072	-1.105	+1.105	-0.541	+0.541

### III. Data and Results

#### Data

The net discount rate is defined as  $ndr_t = i_t - g_t$ , where  $i_t$  is the one-year U.S. treasury bill rate and  $g_t$  is the annual growth rate of average weekly earnings of production workers.<sup>10</sup> Data on these variables are available from 1966 to 2006.

#### Analytic Confidence Boundaries

Equations are used to develop the analytic confidence intervals for each of the four net discount rate estimators. The boundaries are shown in Table 1. The upper and lower boundaries in all cases are symmetric as we assume a normal distribution in calculating the  $z$  statistic. Table 1 shows that with a one-year horizon, for example, the confidence interval on the long-term average estimator is the long-term average plus or minus 2.074 percentage points. In 2006 the long-term average net discount rate was 2.192%.<sup>11</sup> Therefore, the confidence interval contains net discount rates from 0.118% to 4.266%. The other three estimators have slightly smaller one-year horizon confidence boundaries.

For all but the random walk estimator, as the horizon gets longer, the confidence interval narrows. With a 20-year horizon, boundaries for the long-term average and optimal estimator shrink to the current estimate plus or minus about eight-tenths of one percentage point. The compromise estimator has an interval of plus or minus 1.2 percentage points. With a 50-year horizon, the confidence intervals on these three estimators shrink to either one-half or one percentage point, depending on the estimator. In contrast, the random walk

<sup>10</sup>Defining the net discount rate as  $(1+i)/(1+g) - 1$  yields very similar conclusions. The one-year treasury bill rate is taken to be the rate reported in January of each year. The growth rate of earning is measured as a January to next January growth rate.

<sup>11</sup>The long-term average is taken over the entire sample, the years 1966 through 2006.

estimator keeps its boundaries of about two percentage points with almost any horizon.

In their previous paper, Cushing and Rosenbaum (2006) argued that the compromise estimator was a reasonable predictor of the future net discount rate. The compromise estimator equally weighted the current and long-term average net discount rates. The form of the optimal estimator and the time-series properties of net discount rates suggested that this compromise estimator was likely to be reasonably close to the efficient solution. Empirical examples showed that the compromise estimator performed well, significantly outperforming both the long-term average and the current value. They concluded that this compromise estimator provided a simple, transparent method for predicting future net discount rates.

In 2006, the compromise estimator took a value of 2.046. Looking at the boundaries in Table 1, over a 10-year horizon, net discounts would lie in the range 0.72 to 3.37. Over a 20-year horizon, they lie in a range of 0.84 to 3.25. Over 35- and 50-year horizons, the ranges narrow to 0.91-3.18 and 0.94-3.15, respectively.

#### Bootstrap Confidence Boundaries

The process of calculating the bootstrap intervals begins with using observed net discount rates to estimate equation (4) above. The results are shown in equation (7) with standard errors for the coefficient estimates shown in parentheses. Both coefficients are statistically significant at the 95% confidence level. The equation had a standard error of 2.59, an  $R^2$  of 0.292 and a Durbin-Watson statistic of 2.135 suggesting that the first order autocorrelation model fits fairly well.

$$(7) \quad ndr_t = 1.032 + .538 ndr_{t-1} + e_t$$

(.497) (.134)

The estimates of  $\alpha$  and  $\rho$  in equation (7) are then used to create net discount rate estimates over the horizon  $m$ . These are used to calculate distributions as described in “The Model” section above. The resulting confidence interval boundaries are shown in Table 2. The boundaries are not symmetric since they were developed by dropping the top and bottom 25% of observed values, rather than by assuming a symmetric underlying distribution.

With a one-year horizon, the confidence interval on the long-term average estimator is the current value of the long-term average minus 1.613 or plus 1.918 percentage points. The confidence interval would contain net discount rates from 0.579% to 4.110%. The random walk estimator has slightly smaller one-year horizon confidence boundaries. The compromise and optimal estimators have appreciably narrower boundaries.

Table 2  
Bootstrap 50% Confidence Bounds

Horizon (m)	Long-Term Average		Random Walk Estimator		Compromise Estimator		Optimal Estimator	
	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
1	-1.613	1.918	-1.505	1.729	-1.078	1.477	-1.105	1.518
5	-1.392	1.442	-1.972	1.895	-1.421	1.457	-1.327	1.375
10	-1.167	1.243	-2.055	1.877	-1.389	1.326	-1.157	1.178
20	-0.931	0.963	-1.950	1.767	-1.194	1.190	-0.906	0.933
35	-0.813	0.824	-1.948	1.765	-1.165	1.106	-0.805	0.819
50	-0.748	0.759	-1.953	1.709	-1.107	1.046	-0.734	0.754

Table 3  
Bootstrap Relative to Analytic Intervals

Horizon (m)	Long-Term Average		Random Walk Estimator		Compromise Estimator		Optimal Estimator	
	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
1	78%	92%	75%	87%	62%	84%	63%	87%
5	98%	101%	94%	90%	97%	100%	99%	102%
10	106%	113%	97%	89%	105%	100%	108%	110%
20	114%	118%	93%	84%	99%	99%	113%	116%
35	128%	129%	94%	85%	103%	97%	128%	130%
50	137%	139%	94%	82%	100%	95%	136%	139%

For the long-term average estimator, as the horizon gets longer, the confidence interval consistently narrows. With a 20-year horizon, its boundaries are about two-thirds those with a one-year horizon. Over a 50-year horizon, the long-term average estimator's boundaries are less than half the one-year horizon boundaries. The other estimators have boundaries that increase with a 5-year horizon and generally decrease after that.<sup>12</sup> The 50-year horizon boundaries are narrowest for the optimal estimator, followed closely by the long-term average estimator.

Using the compromise estimator and looking at the boundaries in Table 2, over a 10-year horizon, net discounts would lie in the range 0.65 to 3.37. Over a 20-year horizon, they lie in a range of 0.85 to 3.24. Over 35- and 50-year horizons, the ranges narrow to 0.88-3.15 and 0.94-3.09, respectively.

Table 3 compares the bootstrap intervals to the analytic intervals. For the one-year horizon, the bootstrap intervals are from 62% to 92% of the analytic intervals. For all four estimators, the divergence narrows toward one and then

<sup>12</sup>Only the lower bound of the random walk estimator increases from a 5-year to 10-year horizon. That increase is minimal.

widens as the forecast horizon increases. This widening is more pronounced for the long-term average and optimal estimators. For the compromise estimator, the analytic and bootstrap boundaries are very near one another for almost all horizons.

#### Stability of Boundaries over Time

The boundaries in Tables 1 and 2 were calculated using historical data from the years 1966 through 2006 to calculate either variances for the analytic method or the estimators and objective functions for the bootstrap method. This raises the natural question of how the boundaries might vary if the sample period changed for the observed data.

Figure 1 shows how the 10-year horizon confidence intervals vary for the four estimators when the end period for the sample period moves from 1984 to 2005. The upper left hand panel shows 10-year horizon upper and lower confidence bounds for the long-term average estimator. Moving clockwise, the other panels show bounds for the random walk, optimal and compromise estimators. The solid lines represent the bounds for the bootstrap method. The dashed lines represent the bounds for the analytic method. Observation shows that the bounds remain reasonably stable for all four estimators. If anything, they have a tendency to narrow slightly over time. The long-term, optimal and compromise estimators all have bounds of plus or minus about one percentage point. The random walk estimator maintains bounds of plus or minus two percentage points.

#### Impact on Awards

Although confidence intervals on estimates of the net discount rate are interesting in and of themselves, it is also interesting to see how uncertainty translates into the present discounted value of losses. For simplicity, suppose there is a loss of \$1,000,000 spread evenly over  $m$  years. For example, if the loss is to be spread over five years, then the recipient would receive \$200,000 annually. The present discounted value of the payments can then be calculated using the appropriate bootstrap bounds for each of the four estimators. These results are shown in Table 4.

The first row in Table 4 shows the present discounted value of \$200,000 per year paid over five years. The first column for each estimator shows the award when it is discounted at the lowest net discount rate in the 50% confidence interval. The second column for each estimator shows the award when it is discounted at the highest net discount rate in the 50% confidence interval.

With a 5-year horizon, the ranges between the boundaries are not too large, \$75,000 to \$105,000 depending on the estimator. Looking down the columns of Table 4, as the horizon increases, the ranges increase as well. However, the variation between estimators becomes more significant. With a 50-year horizon, for example, the range of values for the long-term estimator is about \$200,000. For the random walk estimator, the range is more than \$500,000.



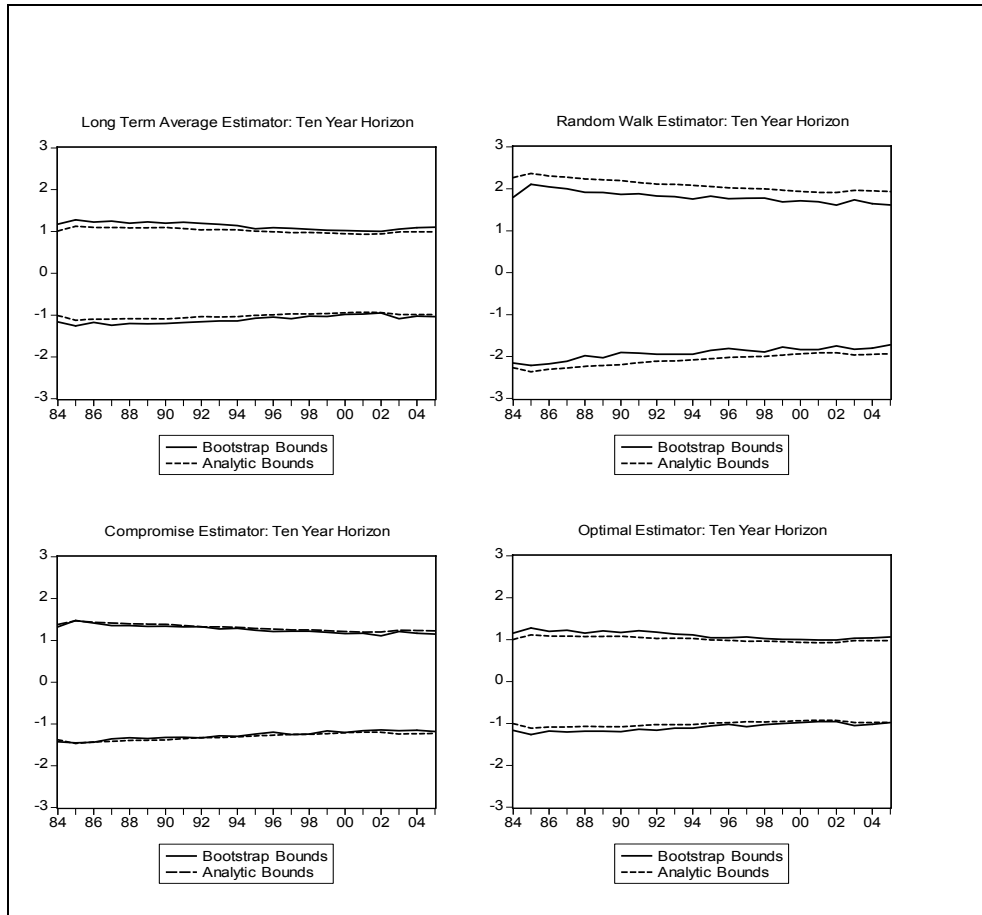


Figure 1  
Variation in Confidence Boundaries

Table 4  
Bootstrap 50% Confidence Bounds on \$1,000,000 Award Paid over m Years

Horizon (m)	Long-term Average		Random Walk Estimator		Compromise Estimator		Optimal Estimator	
	Smallest Net Discount Rate	Largest Net Discount Rate	Smallest Net Discount Rate	Largest Net Discount Rate	Smallest Net Discount Rate	Largest Net Discount Rate	Smallest Net Discount Rate	Largest Net Discount Rate
5	\$984,936	\$907,063	\$1,002,161	\$895,513	\$981,520	\$902,934	\$975,141	\$901,810
10	\$960,834	\$846,824	\$1,008,573	\$820,165	\$964,795	\$837,052	\$946,369	\$837,984
20	\$905,091	\$753,377	\$1,005,259	\$699,995	\$915,866	\$727,899	\$878,565	\$736,703
35	\$828,062	\$639,412	\$1,008,673	\$558,394	\$857,302	\$600,520	\$790,563	\$614,728
50	\$756,933	\$549,635	\$1,013,614	\$460,023	\$795,131	\$505,726	\$709,679	\$521,905

#### IV. Conclusion

Forensic economists typically base their estimates of future net discount rates on the past behavior of the series, using either a long-term average of past rates or simply the current rate as a predictor of future net discount rates. This paper provides analytic and bootstrap 50% confidence intervals for these estimators. In addition we provide confidence intervals for the optimal and compromise estimators derived in an earlier paper.<sup>13</sup> The boundaries for the optimal and compromise estimators are almost always within two percentage points of the estimators, and frequently within one percentage point. In addition, these boundaries have been very stable over time. Combining results from this paper with previous results, a reasonable confidence interval for the net discount rate when losses occur over at least a 10-year horizon would be the compromise estimator plus or minus 1.3 percentage points. In early 2007, that range would be 0.75% to 3.35%.

These confidence intervals may appear surprisingly wide, especially to the applied researcher accustomed to treating the future net discount rate as a known quantity. The time-series behavior of net discount rates indicates that error rates in estimating future values cannot be ignored. These confidence intervals apply to estimates based solely on the past behavior of the series. Alternative approaches to estimating future net discount rates, which may include additional variables, the use of economic theory, "market forecasts" and "expert opinion," could yield more precise estimates. Whether these alternative approaches result in substantially better estimates and hence narrower confidence intervals is the subject of future research.

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<sup>13</sup> See Cushing and Rosenbaum (2006).

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### Cases

Daubert v. Merrell Dow Pharms., Inc., 509 U.S. 579 (1993).

### Appendix

From text equation (4), we have:

$$(A.1) \quad ndr_t = \alpha + \rho ndr_{t-1} + e_t$$

with  $E(e_t) = 0$  and  $\text{Var}(e_t) = \sigma_e^2$ , and  $E(ndr_t) = \frac{\alpha}{1-\rho} = \mu$ . Define  $\widetilde{ndr}_t = ndr_t - \mu$ . Then,

$$(A.2) \quad E(\widetilde{ndr}_t, \widetilde{ndr}_{t-k}) = \frac{\rho^k}{1-\rho^2} \sigma_e^2.$$

Using  $\widehat{PNDR}_t$  as defined in text equation (2),

$$\begin{aligned}
(A.3) \quad E[PNDR_t - \widehat{PNDR}_t]^2 &= E[PNDR_t - (\theta ndr_t + (1-\theta)\mu)]^2 = \\
&E[(PNDR_t - \mu) - \theta(ndr_t - \mu)]^2 = \\
&E\left[\left(\frac{1-\gamma}{1-\gamma^m} \sum_{j=0}^{m-1} \gamma^j \widetilde{ndr}_{t+1+j}\right) - \theta \widetilde{ndr}_t\right]^2
\end{aligned}$$

which, after squaring the terms, equals

$$\begin{aligned}
(A.4) \quad &\left(\frac{1-\gamma}{1-\gamma^m}\right)^2 E\left[\sum_{j=0}^{m-1} \gamma^j \widetilde{ndr}_{t+1+j}\right]^2 - \frac{2\theta(1-\gamma)}{1-\gamma^m} E\left[\widetilde{ndr}_t \sum_{j=0}^{m-1} \gamma^j \widetilde{ndr}_{t+1+j}\right] \\
&+ \theta^2 E\left[\widetilde{ndr}_t\right]^2.
\end{aligned}$$

Focusing on the first term in (A.4), it can be rewritten as:

$$\begin{aligned}
(A.5) \quad &\frac{(1-\gamma)^2 \sigma_e^2}{(1-\gamma^m)^2 (1-\rho^2)} \left[ \sum_{j=0}^{m-1} \sum_{h=0}^{m-1} \gamma^{j+h} \rho^{|j-h|} \right] = \\
&\frac{(1-\gamma)^2 \sigma_e^2}{(1-\gamma^m)^2 (1-\rho^2)} \left[ \sum_{j=0}^{m-1} \sum_{h=0}^{j-1} \gamma^{j+h} \rho^{|j-h|} + \sum_{j=0}^{m-1} \sum_{h=j}^{m-1} \gamma^{j+h} \rho^{|j-h|} + \sum_{j=0}^{m-1} \sum_{h=j+1}^{m-1} \gamma^{j+h} \rho^{|j-h|} \right].
\end{aligned}$$

Noting that in (A.5), the first and third double sums are equal, (A.5) becomes

$$(A.6) \quad \frac{(1-\gamma)^2 \sigma_e^2}{(1-\gamma^m)^2 (1-\rho^2)} \left[ \sum_{j=0}^{m-1} \left( \gamma^{2j} + 2 \sum_{h=j+1}^{m-1} \gamma^{j+h} \rho^{|j-h|} \right) \right].$$

Substituting  $k$  for  $h-j$ , (A.6) becomes

$$(A.7) \quad \Omega \left[ \sum_{j=0}^{m-1} \left( \gamma^{2j} + 2 \sum_{k=1}^{m-1-j} \gamma^{2j+k} \rho^{|k|} \right) \right],$$

where  $\Omega = \frac{(1-\gamma)^2 \sigma_e^2}{(1-\gamma^m)^2 (1-\rho^2)}$ . Now, (A.7) can be expressed as

$$\begin{aligned}
(A.8) \quad &\Omega \left[ \sum_{j=0}^{m-1} \left( \gamma^{2j} \left( 1 + 2 \sum_{k=1}^{m-1-j} \gamma^k \rho^{|k|} \right) \right) \right] = \\
&\Omega \left[ \sum_{j=0}^{m-1} \left( \gamma^{2j} \left( 2 \sum_{k=0}^{m-1-j} (\gamma\rho)^k - 1 \right) \right) \right] = \\
&\Omega \left[ \sum_{j=0}^{m-1} \left( \gamma^{2j} \left( 2 \frac{1 - (\gamma\rho)^{m-j}}{1 - \gamma\rho} - 1 \right) \right) \right] = \\
&\Omega \left[ \frac{(1 + \gamma\rho)(1 - \gamma^{2m})}{(1 - \gamma\rho)(1 - \gamma^2)} - \sum_{j=0}^{m-1} \frac{2\gamma^m \rho^m}{1 - \gamma\rho} (\gamma/\rho)^j \right].
\end{aligned}$$

Carrying out the summation and substituting back for  $\Omega$  provides:

$$(A.9) \quad \left[ \frac{(1-\gamma)^2 \sigma_e^2}{(1-\gamma^m)(1-\rho^2)} \right] \left[ \frac{(1+\gamma\rho)(1-\gamma^{2m})}{(1-\gamma\rho)(1-\gamma^2)} - \left( \frac{2\gamma^m \rho^m}{1-\gamma\rho} \frac{1-(\gamma/\rho)^m}{(1-\gamma/\rho)} \right) \right].$$

This is the last term in text equation (5).

Now turn to the second term in (A.4). Recalling (A.2), this term becomes:

$$(A.10) \quad \frac{-2\theta(1-\gamma)}{1-\gamma^m} \sum_{j=0}^{m-1} \frac{\gamma^j \rho^{j+1}}{1-\rho^2} \sigma_e^2 =$$

$$\frac{-2\theta(1-\gamma)\rho\sigma_e^2}{(1-\gamma^m)(1-\rho^2)} \sum_{j=0}^{m-1} (\gamma\rho)^j,$$

or

$$(A.11) \quad \sigma_e^2 \frac{-2\rho(1-\gamma)(1-(\gamma\rho)^m)}{(1-\gamma\rho)(1-\rho^2)(1-\gamma^m)} \theta,$$

which is the middle term in text equation (5). Finally, using (A.2), the last term in (A.4) becomes:

$$(A.12) \quad \frac{\theta^2 \sigma_e^2}{1-\rho^2}.$$